Smooth parametric dependence of asymptotics of the semiclassical focusing NLS

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Abstract

We consider the one dimensional focusing (cubic) Nonlinear Schrödinger equation (NLS) in the semiclassical limit with a one parameter family of exponentially decaying in absolute value initial conditions. We prove smooth parametric dependence of the asymptotic solution on the parameter utilizing the Riemann-Hilbert approach. Numerical computations supporting our estimates of important quantities are presented.

1 Introduction

We study a type of scalar Riemann-Hilbert problem that appears in semiclassical or small dispersion limit of integrable systems. We prove the smooth dependence of the solution on a crucial parameter. Our motivation comes from the semiclassical focusing NLS equation [2, 16, 17, 18]

$$i\varepsilon\partial_t q + \varepsilon^2\partial_x^2 q + 2|q|^2 q = 0 \tag{1}$$

with the initial condition

$$q(x,0) = A(x)e^{\frac{i\mu}{\varepsilon}S(x)}, \quad A(x) > 0, \quad \mu \geq 0 \tag{2}$$

in the limit as $\varepsilon \to 0$, where the parameter μ is the focus of this paper. For any value of $\varepsilon > 0$, the solution process requires solving a 2×2 matrix Riemann-Hilbert problem (RHP) in the complex plane of the spectral parameter z of an underlying Lax pair operator [23, 24]. The quantities x, t, and μ enter as parameters. The Riemann-Hilbert approach is used to other integrable systems in general as established by [19] and it is a major tool in the asymptotic analysis of integrable systems as established with the discovery of the steepest descent method [10, 11]. The asymptotic methods via the RHP approach also apply to orthogonal polynomial asymptotics [6, 7], and to random matrices [1, 8, 12, 13].

Riemann-Hilbert problems formalism is another form of the (2D) potential theory in the complex plane of the spectral parameter z. The main question is to find the equilibrium measure of a set - a minimizer of an energy functional [6, 20]. This equilibrium measure is the spectral measure of the underlying linear operator. In the case of NLS, it is (non-selfadjoint)

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Zakharov-Shabat system (a Dirac-type operator). For self-adjoint operators the equilibrium measure is supported on a subset of the real axis. In a general non-self-adjoint case the support is a subset of the complex plane. A typical support of the equilibrium measure in the case of NLS in the semiclassical limit seems to be a finite union of arcs in the complex plane with complex conjugate symmetry [4, 5, 16, 23, 25]. Denote the end points of the arcs as $\{\alpha_j\}_{j=0}^{N'}$ with some finite $N' \in \mathbb{N}$. This leads to finite genus Riemann surfaces in the asymptotic analysis.

There are several approaches to obtaining asymptotic solution from the initial data. One approach is to analyze the RHP for a fixed x and t. Another approach is to look at the dependence of the RHP on x and t as parameters. This includes the analysis of the contributing limiting arcs through analysis of the end points α_j. These are often referred as branchpoints since they arise as the branchpoints of a square root.

The smooth dependence of the branchpoints α_j with respect to parameters x and t was obtained in [21] through formulae of the form

$$\frac{\partial \alpha_j}{\partial x}(x,t) = -\frac{2\pi i \frac{\partial K}{\partial x}(\alpha_j, \vec{\alpha}, x, t)}{D(\vec{\alpha}, x, t) \oint_{\hat{\gamma}} \frac{f'(\zeta,x,t)}{(\zeta-\alpha_j(x,t))R(\zeta,\vec{\alpha})} d\zeta}, \tag{3}$$

$$\frac{\partial \alpha_j}{\partial t}(x,t) = -\frac{2\pi i \frac{\partial K}{\partial t}(\alpha_j, \vec{\alpha}, x, t)}{D(\vec{\alpha}, x, t) \oint_{\hat{\gamma}} \frac{f'(\zeta,x,t)}{(\zeta-\alpha_j(x,t))R(\zeta,\vec{\alpha})} d\zeta}. \tag{4}$$

We consider initial data being a semiclassical approximation [23] of

$$q(x,0) = -\text{sech }(x) e^{-\frac{i\mu}{\varepsilon}\int_0^x \tanh(s)ds}, \quad \mu \geq 0. \tag{5}$$

This family of initial conditions is interesting because of a transition at $\mu = 2$ with solitonless interval ($\mu \geq 2$) and radiation with solitons interval $0 < \mu < 2$. It has been studied in a number of papers [23, 24, 25]. The semiclassical limit has been completely analyzed for $\mu \geq 2$ while for the soliton case $0 < \mu < 2$ the answer is known only for some finite times. The RH approach runs into difficulties with the error estimates and was not able to continue past a certain curve. Numerical experiments have shown absence of any noticeable transition in the behavior of the arcs end points $\alpha_j(\mu)$ at the critical value $\mu = 2$ [3]. The perturbation theorem for NLS 3.6 establishes this fact rigorously.

We extend formulae (3-4) to the case of the contour explicitly depending on an external parameter μ, $\hat{\gamma} = \hat{\gamma}(\mu)$

$$\frac{\partial \alpha_j}{\partial \mu}(x,t,\mu) = -\frac{2\pi i \frac{\partial K}{\partial \mu}(\alpha_j, \vec{\alpha}, x, t, \mu)}{D(\vec{\alpha}, x, t, \mu) \oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta,x,t,\mu)}{(\zeta-\alpha_j(\mu))R(\zeta,\vec{\alpha})} d\zeta} \tag{6}$$

and the dependence is smooth, meaning that the contour, the jump matrix, and the solution of the scalar RHP evolve smoothly in μ. Moreover we simplify the expression for $\frac{\partial K}{\partial \mu}$ as (37). We also compare (see Fig. 3) formula (6) with the direct computations of the branchpoints from solving the system (13).

Moreover, we prove the preservation of genus of the solution for all μ in an open interval around any μ_0 for which the genus is known. In particular, the genus is preserved (0 or 2) for all x and $t > 0$ for some open interval (which depends on x and t) for $\mu < 2$.

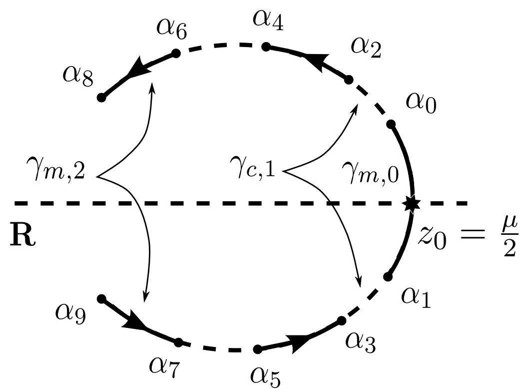


Figure 1: The RHP jump contour in the case of genus 4 with complex conjugate symmetry in the notation of [23].

The paper is organized as the following: Section 2 - main definitions and prior results are stated, Section 3.1 - we prove analyticity of f in μ, which leads to differentiability of the branchpoints $\alpha_j = \alpha_j(\mu)$, Section 3.2 - we consider μ dependence of all main quantities stated in theorem 3.6, Section 3.3 - we additionally consider sign conditions and prove preservation of genus in theorem 3.15.

2 Preliminaries

We consider the scalar Riemann-Hilbert problem (RHP) which arises as a model (simplified) RHP in the process of asymptotic solution of the semiclassical focusing NLS (1) with the initial condition (5)

$$\begin{cases} h_+(z) + h_-(z) = 2W_j, \text{ on } \gamma_{m,j}, \quad j = 0, 1, ..., N, \\ h_+(z) - h_-(z) = 2\Omega_j, \text{ on } \gamma_{c,j}, \quad j = 1, ..., N, \\ h(z) + f(z) \text{ is analytic in } \overline{\mathbb{C}} \backslash \gamma, \end{cases} \tag{7}$$

where the jump contour γ belongs to the class of admissible contours with the following properties.

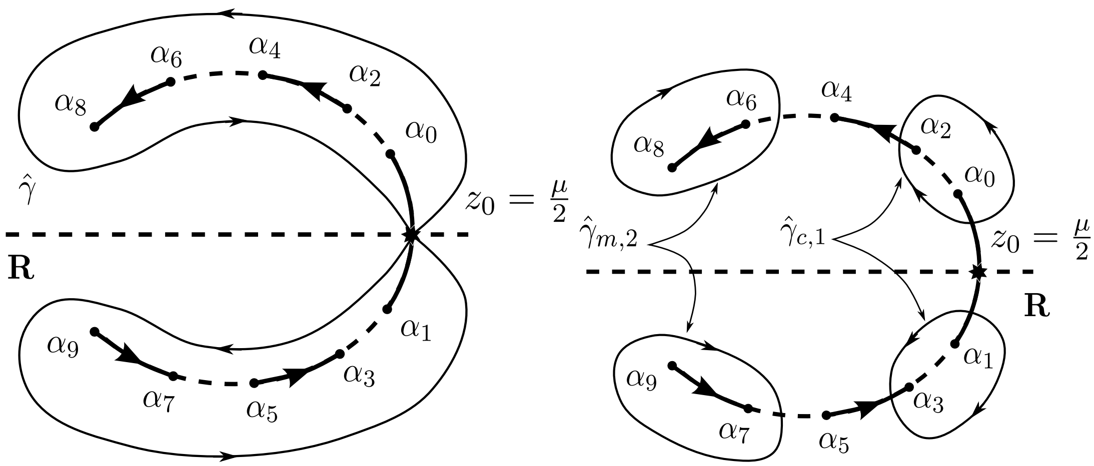
Definition 2.1.
We say

$$\gamma \in \Gamma(\vec{\alpha}, \mu)$$

if a simple contour γ consists of a finite union of oriented finite length piecewise continuous simple arcs in the complex plane $\gamma = (\cup \gamma_{m,j}) \cup (\cup \gamma_{c,j})$ with the distinct arcs end points $\vec{\alpha} = \{\alpha_j\}_{j=0}^{4N+1}$, as shown in Fig. 1. We denote the main $\gamma_{m,j}$ and the complementary arcs $\gamma_{c,j}$ as the following

$$\gamma_{m,0} = [\alpha_1, \alpha_0], \quad \gamma_{m,j} = [\alpha_{4j-2}, \alpha_{4j}] \cup [\alpha_{4j+1}, \alpha_{4j-1}], \quad j = 1, ..., N,$$

$$\gamma_{c,j} = [\alpha_{4j-4}, \alpha_{4j-2}] \cup [\alpha_{4j-1}, \alpha_{4j-3}], \quad j = 1, ..., N.$$



(a) Contour $\hat{\gamma}$ (b) Contours $\hat{\gamma}_{m,j}$

Figure 2: Contours of integration for function $h(z)$ (8). z_0 is a point of non-analyticity of $f(z)$ on γ, and $z_0 = \frac{\mu}{2}$ depends of an external parameter μ.

Let $z_0 = \frac{\mu}{2} \in \gamma$, be a point of logarithmic singularity of f.

For simplicity we assume that γ is not tangent to the real axis at $z = \frac{\mu}{2} \geq 0$. We additionally assume that the contour γ has complex conjugate symmetry $\overline{\gamma} = \gamma$.

Note that for fixed $\vec{\alpha}$ and μ the contour γ aside from passing through $z = \alpha_j$ and $z = \frac{\mu}{2}$ is free to deform continuously within of domain of analyticity of f. Thus for a fixed f, a contour $\gamma \in \Gamma(\vec{\alpha}, \mu)$ depends on the arcs end points $\vec{\alpha}$ and on μ through z_0, $\gamma = \gamma(\vec{\alpha}, \mu)$.

An immediate fact about these contours is the following.

Lemma 2.2. *Let $\gamma_0 = \gamma_0(\vec{\alpha}_0, \mu_0) \in \Gamma(\vec{\alpha}_0, \mu_0)$ with the distinct arcs end points $\vec{\alpha}_0$. Then there exist open neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ in the neighborhood of $\vec{\alpha}_0$, for all μ in the neighborhood of μ_0 there is a contour $\gamma(\vec{\alpha}, \mu) \in \Gamma(\vec{\alpha}, \mu)$.*

Definition 2.3.
We say

$$\hat{\gamma} \in \hat{\Gamma}(\gamma, \vec{\alpha}, \mu)$$

if $\hat{\gamma}$ is a simple closed contour around $\gamma \in \Gamma(\vec{\alpha}, \mu)$ within the domain of analyticity of f. with complex conjugate symmetry $\overline{\hat{\gamma}} = \hat{\gamma}$.

Remark 2.4. Similarly we define $\hat{\gamma}_{m,j}$ and $\hat{\gamma}_{c,j}$.

Remark 2.5. By considering the loop contours $\hat{\gamma}$, $\hat{\gamma}_{m,j}$, $\hat{\gamma}_{c,j}$, the explicit dependence of the contours on the end points $\vec{\alpha}$ is removed (for example in (29-32)). So even though $\gamma = \gamma(\vec{\alpha}, \mu)$, in all our evaluations below $\hat{\gamma} = \hat{\gamma}(\mu)$.

Remark 2.6. Lemma 2.2 implies that if $\hat{\gamma}_0 \in \hat{\Gamma}(\gamma_0, \vec{\alpha}_0, \mu_0)$ then there is a contour $\gamma \in \Gamma(\vec{\alpha}, \mu)$ such that $\hat{\gamma} \in \hat{\Gamma}(\gamma, \vec{\alpha}, \mu)$ for all $\vec{\alpha}$ and μ in some open neighborhoods of $\vec{\alpha}_0$ and μ_0.

The solution of the RHP (7), $h(z)$ can be found explicitly [23]

$$h(z)=\frac{R(z)}{2\pi i}\left[\oint_{\hat\gamma}\frac{f(\zeta)}{(\zeta-z)R(\zeta)}d\zeta+\sum_{j=0}^{N}\oint_{\hat\gamma_{m,j}}\frac{W_j}{(\zeta-z)R(\zeta)}d\zeta+\sum_{j=1}^{N}\oint_{\hat\gamma_{c,j}}\frac{\Omega_j}{(\zeta-z)R(\zeta)}d\zeta\right], \quad (8)$$

or in the determinant form [21]

$$h(z)=\frac{R(z)}{D}K(z), \quad (9)$$

where z lies inside of $\hat\gamma$ and outside all $\hat\gamma_{c,j}$ and $\hat\gamma_{m,j}$, and where

$$K(z)=\frac{1}{2\pi i}\begin{vmatrix}\oint_{\hat\gamma_{m,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \dots & \dots & \dots & \dots\\ \oint_{\hat\gamma_{m,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{m,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \dots & \dots & \dots & \dots\\ \oint_{\hat\gamma_{c,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{c,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,N}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \oint_{\hat\gamma}\frac{f(\zeta)d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma}\frac{\zeta^{N-1}f(\zeta)d\zeta}{R(\zeta)} & \oint_{\hat\gamma}\frac{f(\zeta)d\zeta}{(\zeta-z)R(\zeta)}\end{vmatrix} \quad (10)$$

and

$$D=det(A), \quad (11)$$

with

$$A=\begin{pmatrix}\oint_{\hat\gamma_{m,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \dots & \dots & \dots\\ \oint_{\hat\gamma_{m,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{m,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \dots & \dots & \dots\\ \oint_{\hat\gamma_{c,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat\gamma_{c,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\end{pmatrix}. \quad (12)$$

The arcs end points $\{\alpha_j\}$ satisfy the system

$$K(\alpha_j)=0, \quad j=0,1,\dots,4N+1. \quad (13)$$

The dependence on x and t was considered in [21]. This is a simpler situation when the jump contour γ in the RHP (7) is independent of the parameters.

The main related results in [21] are the determinant form (9) and

Theorem 2.7. *Let $f(z,\vec\beta)$, where $\vec\beta\in B\subset\mathbb{R}^m$. For all $\vec\beta\in B$ assume $f(z,\vec\beta)$ be analytic in on $S\in\mathbb{C}$. Moreover, $\gamma\backslash S$ consists of no more than finitely points and f continuous on γ. The modulation equations (13) imply the system of $4N+2$ differential equations*

$$(\alpha_j)_{\beta_k}=-\frac{2\pi i\frac{\partial}{\partial\beta_k}K(\alpha_j)}{D\oint_{\hat\gamma}\frac{f'(\zeta)}{(\zeta-\alpha_j)R(\zeta)}d\zeta} \quad (14)$$

In particular, one gets (3) and (4) for parameters x and t. Note, that the contour γ is assumed independent of parameters x and t explicitly. The dependence on these parameters comes in through the branchpoints $\vec{\alpha} = \vec{\alpha}(x,t)$.

The main related result in [22] is

Theorem 2.8. *Let the nonlinear steepest descent asymptotics for solution $q(x,t,\varepsilon)$ of the NLS (1) be valid at some point (x_0,t_0). If (x_*,t_*) is an arbitrary point, connected with (x_0,t_0) by a piecewise-smooth path Σ, if the contour $\gamma(x,t)$ of the RHP (7) does not interact with singularities of $f(z)$ as (x,t) varies from (x_0,t_0) to (x_*,t_*) along Σ, and if all the branchpoints are bounded and stay away from the real axis, then the nonlinear steepest descent asymptotics (with the proper choice of the genus) is also valid at (x_*,t_*).*

We extend Theorem 2.7 and make partial progress in the direction of Theorem 2.8 in the case when the jump contour explicitly depends on the parameter μ. We require that the point of logarithmic singularity of $f = f(z,\mu)$, $z_0 = \frac{\mu}{2}$ is always on γ. Additionally we prove preservation of genus for all $x > 0$, $t > 0$, $\mu > 0$, under certain conditions which guarantee that the parameters are away from the asymptotic solution break Theorem 3.15. In particular, the genus is preserved in the neighborhood of the special value of the parameter $\mu = 2$. Thus we obtain that for all $x > 0$, $t > 0$ (except on the first breaking curve) there is a small neighborhood such that for all $\mu < 2$ in the neighborhood, the genus is the same as for $\mu = 2$, where it is known to be 0 or 2.

3 μ-dependence in the semiclassical focusing NLS

3.1 Setup

To apply the methods from [21] we need analyticity of $f(z,\mu)$ in the parameter μ.

The function $f(z)$ comes from a semiclassical approximation of the family of initial conditions for NLS (5) as [23]:

$$f(z,\mu,x,t) = \left(\frac{\mu}{2} - z\right)\left[\frac{\pi i}{2} + \ln\left(\frac{\mu}{2} - z\right)\right] + \frac{z+T}{2}\ln(z+T) + \frac{z-T}{2}\ln(z-T)$$
$$- T\tanh^{-1}\frac{T}{\mu/2} - xz - 2tz^2 + \frac{\mu}{2}\ln 2, \quad \text{when } \Im z \geq 0 \tag{15}$$

and

$$f(z) = \overline{f(\overline{z})}, \quad \text{when } \Im z < 0, \tag{16}$$

where the branchcuts are chosen as the following: for $0 < \mu < 2$ the logarithmic branchcut is from $\frac{\mu}{2}$ along the real axis to $+\infty$, from T to 0 and along the real axis to $+\infty$, from $-T$ to 0 and along the real axis to $-\infty$; for $\mu \geq 2$ - from T to $+\infty$ and from $-T$ to $-\infty$ along the real axis, where

$$T = T(\mu) = \sqrt{\frac{\mu^2}{4} - 1}, \quad \Im T \geq 0. \tag{17}$$

For $\mu \geq 2$, $T \geq 0$ is real and for $0 < \mu < 2$, T is purely imaginary with $\Im T > 0$.

Then for $\Im z > 0$

$$\frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln\left(\frac{\mu}{2} - z\right) + \ln 2 + \frac{\mu}{8T}\left[\ln(z+T) - \ln(z-T) - 2\tanh^{-1}\frac{2T}{\mu}\right] \tag{18}$$

where $\tanh^{-1} x = x + O(x^3),\ x \to 0$ then for $\Im z \neq 0$

$$\frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln\left(\frac{\mu}{2} - z\right) + \ln 2 + \frac{\mu}{4z} - \frac{1}{2} + O(T),\ T \to 0. \tag{19}$$

So $\mu = 2$ is a removable singularity for $f_\mu(z,\mu)$ and

$$\lim_{\mu\to 2, T\to 0} \frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln(1-z) + \ln 2 + \frac{1}{2z} - \frac{1}{2}, \tag{20}$$

which is analytic in z for $\Im z \neq 0$.

Lemma 3.1. *$f(z,\mu)$ and $f'(z,\mu)$ are analytic in μ for $\mu > 0$, $x > 0$, $t > 0$, for all z, $\Im z \neq 0$, $z \notin [-T,T]$.*

Proof. Consider

$$f'(z,\mu) = -\frac{\pi i}{2} - \ln\left(\frac{\mu}{2} - z\right) + \frac{1}{2}\ln\left(z^2 - \frac{\mu^2}{4} + 1\right) - x - 4tz, \tag{21}$$

which analytic in $\mu > 0$, for $\Im z \neq 0$, $z \notin [-T,T]$.

For $\mu > 0$, $\mu \neq 2$, $f(z,\mu)$ is clearly analytic in μ for $\Im z \neq 0$. At $\mu = 2$ ($T = 0$) we find the power series of $f(z,\mu)$ in T and show that it contains only even powers. Since

$$T^{2k} = \left(\frac{\mu^2}{4} - 1\right)^k = \frac{(\mu+2)^k}{4^k}(\mu-2)^k \tag{22}$$

it will show analyticity of $f(z,\mu)$ in μ.

Start with expanding in series at $T = 0$

$$\frac{1}{\mu/2} = \sqrt{1+T^2}^{-1} = \sum_{k=0}^{\infty} c_k T^{2k}, \quad \ln(z \pm T) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\pm\frac{T}{z}\right)^n. \tag{23}$$

Then the terms in (15) are analytic in μ

$$\frac{z+T}{2}\ln(z+T) + \frac{z-T}{2}\ln(z-T) \tag{24}$$

$$= z\ln z - z\sum_{n\ is\ even}\frac{1}{n}\left(\frac{T}{z}\right)^n + T\sum_{n\ is\ odd}\frac{1}{n}\left(\frac{T}{z}\right)^n \tag{25}$$

$$= z\ln z + \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)z^{2k-1}}T^{2k}, \tag{26}$$

which has only even powers of T and is analytic in μ for $\Im z \neq 0$. The inverse hyperbolic tangent term in (15) and taking into account that $\tanh^{-1} z$ is an odd function

$$T\tanh^{-1}\frac{T}{\mu/2} = T\tanh^{-1}\frac{T}{\sqrt{1+T^2}} = T\tanh^{-1}\sum_{k=0}^{\infty} c_k T^{2k+1} \tag{27}$$

$$= T\sum_{k=0}^{\infty} \tilde{c}_k T^{2k+1} = \sum_{k=0}^{\infty} \tilde{c}_k T^{2k+2}, \tag{28}$$

which also has only even powers of T.

So $f(z, x, t, \mu)$ is analytic in μ for $\mu > 0$, $x > 0$, $t > 0$, $\Im z \neq 0$, $z \notin [-T(\mu), T(\mu)]$.

□

Note the jump of $f(z)$ is caused by the Schwarz reflection (16) on the real axis and it is linear in z since $\Im f$ is a linear function on the real axis (as a limit) near $\frac{\mu}{2}$ with $\Im f\left(\frac{\mu}{2}\right) = 0$ [23].

3.2 Parametric dependence of scalar RHP

The main difficulty is the dependence of $f(z)$ (thus the RHP (7)) and the modulation equations (13) on parameter μ which also controls the logarithmic branchpoint $z = \frac{\mu}{2}$ on the contour $\hat{\gamma}$. We show that the dependence on μ is smooth.

To solve $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$, we need the Implicit function theorem and nondegeneracy of the system. The following technical lemma simplifies the expressions (8), (10).

Lemma 3.2.

Let function f be given by (15) and there is a contour $\gamma_0 = \gamma(\vec{\alpha}, \mu_0) \in \Gamma(\vec{\alpha}, \mu_0)$ has fixed arcs end points $\vec{\alpha}$. Then there is an open neighborhood of μ_0 such that for all μ in a small open neighborhood of μ_0 there is $\hat{\gamma}(\mu) \in \hat{\Gamma}(\gamma, \vec{\alpha}, \mu)$

$$\frac{\partial}{\partial\mu}\oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta, \mu) d\zeta}{R(\zeta, \vec{\alpha})} = \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n \frac{\partial f(\zeta,\mu)}{\partial\mu} d\zeta}{R(\zeta, \vec{\alpha})}, \quad n \in \mathbb{N}, \tag{29}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat{\gamma}(\mu)} \frac{f(\zeta, \mu) d\zeta}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})} = \oint_{\hat{\gamma}(\mu)} \frac{\frac{\partial f(\zeta,\mu)}{\partial\mu} d\zeta}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})}, \quad n \in \mathbb{N}, \tag{30}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat{\gamma}_{m,j}} \frac{\zeta^n d\zeta}{R(\zeta, \vec{\alpha})} = 0, \quad \frac{\partial}{\partial\mu}\oint_{\hat{\gamma}_{m,j}} \frac{d\zeta}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})} = 0, \quad j = 1, 2, \ldots, N, \ n \in \mathbb{N}, \tag{31}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat{\gamma}_{c,j}} \frac{\zeta^n d\zeta}{R(\zeta, \vec{\alpha})} = 0, \quad \frac{\partial}{\partial\mu}\oint_{\hat{\gamma}_{c,j}} \frac{d\zeta}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})} = 0, \quad j = 1, 2, \ldots, N, \ n \in \mathbb{N}. \tag{32}$$

Proof.

The idea of the proof is to consider finite differences and take the limit as $\Delta\mu \to 0$. The complication is that both the integrands and the contours of integration depend on μ.

Denote the integral on the left in (29) as I_1

$$I_1(\mu) = \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu)}{R(\zeta,\vec{\alpha})} d\zeta. \tag{33}$$

where $\hat{\gamma}(\mu) \in \hat{\Gamma}(\gamma,\vec{\alpha},\mu)$. Consider

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu},$$

with small real $\Delta\mu \neq 0$. There are two logarithmic branchcuts near the contours of integration: in $f(z,\mu)$ and in $f(z,\mu+\Delta\mu)$ with both branchcuts are chosen from $z_0(\mu) := \frac{\mu}{2}$ and $z_0(\mu+\Delta\mu)$ horizontally to the right along the real axis. Additionally, these functions have a jump on the real axis for $z < \frac{\mu}{2}$ from Schwarz symmetry.

We choose some fixed points δ_1 and δ_2 to be real, $\delta_1 < \frac{\mu}{2} - \frac{|\Delta\mu|}{2} < \frac{\mu}{2} + \frac{|\Delta\mu|}{2} < \delta_2$. Both contours of integration $\hat{\gamma}(\mu)$, $\hat{\gamma}(\mu+\Delta\mu)$ are pushed to the real axis near z_0 and split into $[\delta_1,\delta_2] := [\delta_1+i0,\delta_2+i0]\bigcup[\delta_2-i0,\delta_1-i0]$ and its complement. On the complement, we can also deform both contours to coincide. So $\hat{\gamma}(\mu+\Delta\mu) = \hat{\gamma}(\mu)$.

Note that across $[\delta_1,\delta_2]$, $f(z,\mu)$ has a jump $\pi i\,|z_0(\mu)-z|$ and $f(z,\mu+\Delta\mu)$ has a jump $\pi i\,|z_0(\mu+\Delta\mu)-z|$. So contributions near z_0 in both cases are small.

Then

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu} = \frac{1}{\Delta\mu}\left(\oint_{\hat{\gamma}(\mu+\Delta\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})} d\zeta - \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu)}{R(\zeta,\vec{\alpha})} d\zeta\right) \tag{34}$$

we add and subtract $\oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})} d\zeta$

$$= \frac{1}{\Delta\mu}\left(\oint_{\hat{\gamma}(\mu+\Delta\mu)-\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})} d\zeta + \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n (f(\zeta,\mu+\Delta\mu) - f(\zeta,\mu))}{R(\zeta,\vec{\alpha})} d\zeta\right) \tag{35}$$

The first integral is 0, because $\hat{\gamma}(\mu+\Delta\mu) = \hat{\gamma}(\mu)$.

Thus

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu} = \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n \frac{f(\zeta,\mu+\Delta\mu) - f(\zeta,\mu)}{\Delta\mu}}{R(\zeta,\vec{\alpha})} d\zeta. \tag{36}$$

The last step is to take the limit as $\Delta\mu \to 0$ and to interchange it with the integral. The contour of integration is split into two: a small neighborhood near z_0 and its complement. For the integral near z_0, by a direct computation can be shown that the limit can be passed under the integral. The integral over the second part of the contour has the integrand uniformly bounded in μ since $\log\left(\zeta - \frac{\mu}{2}\right)$ in $\frac{\partial f}{\partial\mu}$ is uniformly bounded away from $\frac{\mu}{2}$, so the limit and the integral can be interchanged. This completes the proof for the first integral (29).

The second integral (30) is done similarly. The rest of the integrals (31)-(32) are independent of μ since the only dependence on μ sits in $z_0(\mu) \in \gamma_{m,0}$.

$\square$

Using lemma 3.2,

$$\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \dots & \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{m,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \dots & \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{c,n}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,n}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}}\frac{\zeta^{N-1}f_\mu(\zeta)d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)d\zeta}{(\zeta-\alpha_j)R(\zeta)} \end{vmatrix}, \tag{37}$$

where f_μ is given by (18).

Lemma 3.3.
Let f be given by (15) and the contour $\gamma_0=\gamma(\vec{\alpha}_0,\mu_0)\in\Gamma(\vec{\alpha}_0,\mu_0)$ has the arcs end points $\vec{\alpha}_0$. Then there are open neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ and μ in the neighborhoods of $\vec{\alpha}_0$ and μ_0 respectively there is a contour $\gamma=\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$ and

$$K_j(\vec{\alpha},\mu):=K(\alpha_j,\vec{\alpha},\mu),\qquad j=0,1,\dots,4N+1 \tag{38}$$

is continuously differentiable in $\vec{\alpha}$ and in μ.

Proof.
First we notice that since $\gamma_0\in\Gamma(\vec{\alpha}_0,\mu_0)$ by Lemma 2.2, there are neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ and μ in the neighborhoods of $\vec{\alpha}_0$ and μ_0 respectively there is a contour $\gamma=\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$.

$K_j(\vec{\alpha},\mu)$ is analytic in $\vec{\alpha}$ by the determinant structure and the integral entries (10), where explicit dependence on $\vec{\alpha}$ is only in the $R(z,\vec{\alpha})$ term which is analytic away from $z=\alpha_j$, $j=0,...,4N+1$.

The integrals in the last row of $\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)$ in (37) involve the function f_μ given by (18) which is integrable near $z=\frac{\mu}{2}$ and hence $\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)$ is continuous in μ. Thus $K_j(\alpha_j,\vec{\alpha},\mu)$ is continuously differentiable in $\vec{\alpha}$ and in μ.

□

By Lemma 3.3 the modulation equations (13)

$$K_j(\vec{\alpha},\mu)=K(\alpha_j,\vec{\alpha},\mu)=0 \tag{39}$$

are smooth in $\vec{\alpha}$ and in the parameter μ. Next we want to solve this system for $\vec{\alpha}=\vec{\alpha}(\mu)$ and conclude smoothness in μ by the Implicit function theorem.

For the next lemma we need $K'(z,\vec{\alpha},\mu)=\frac{dK}{dz}(z,\vec{\alpha},\mu)$

$$K'(z,\vec{\alpha},\mu)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \dots & \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \dots & \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f(\zeta)d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}}\frac{\zeta^{N-1}f(\zeta)d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f(\zeta)d\zeta}{(\zeta-z)^2R(\zeta)} \end{vmatrix}, \tag{40}$$

where z is inside of $\hat{\gamma}(\mu)$ and inside of $\hat{\gamma}_{m,j}$ and $\hat{\gamma}_{c,j}$ or $\hat{\gamma}_{c,j+1}$.

Lemma 3.4.
Let f be given by (15) and a contour $\gamma_0 \in \Gamma(\vec{\alpha}_0, \mu_0)$, where $\vec{\alpha}_0$ and μ_0 satisfy

$$\vec{K}(\vec{\alpha}_0, \mu_0) = \vec{0}.$$

Assume that for $\vec{\alpha}_0 = \left\{\alpha_j^0\right\}_{j=0}^{4N+1}$, $\lim_{z\to\alpha_j^0} K'(z, \vec{\alpha}_0, \mu_0) \neq 0,\ j = 0, 1, \ldots, 4N+1$.

Then the modulation equations

$$\vec{K}(\vec{\alpha}, \mu) = \vec{0}$$

can be uniquely solved for $\vec{\alpha} = \vec{\alpha}(\mu)$ which is continuously differentiable for all μ in some open neighborhood of μ_0 and $\vec{\alpha}(\mu_0) = \vec{\alpha}_0$.

Proof. $\vec{K}$ is continuously differentiable in $\vec{\alpha}$ and in μ by Lemma 3.3.

As it was shown in [21], the matrix $\left\{\frac{\partial \vec{K}}{\partial \vec{\alpha}}\right\}_{j,l} = \left\{\frac{\partial K(\alpha_j)}{\partial \alpha_l}\right\}_{j,l}$ is diagonal and

$$\frac{\partial K(\alpha_j)}{\partial \alpha_j} = \frac{3}{2} D \lim_{z\to\alpha_j^0} \left(\frac{h(z)}{R(z)}\right)' = \frac{3}{2} \lim_{z\to\alpha_j^0} K'(z, \vec{\alpha}, \mu) \neq 0. \tag{41}$$

So

$$\det \left| \frac{\partial \vec{K}}{\partial \vec{\alpha}}(\vec{\alpha}_0) \right| = \prod_j \frac{\partial K(\alpha_j)}{\partial \alpha_j} \neq 0 \tag{42}$$

under the assumptions. By the Implicit function theorem, $\vec{\alpha}(\mu)$ are uniquely defined in some neighborhood of μ_0 and smooth in μ. Note $\vec{\alpha}(\mu_0) = \vec{\alpha}_0$ by assumption. □

Remark 3.5.
The condition $\lim_{z\to\alpha_j^0} K'(z, \vec{\alpha}_0, \mu_0) \neq 0,\ j = 0, 1, \ldots, 4N+1$ in Lemma 3.4 is equivalent to $\lim_{z\to\alpha_j^0} \frac{h'(z,\vec{\alpha}_0,\mu_0)}{R(z,\vec{\alpha}_0)} \neq 0,\ j = 0, 1, \ldots, 4N+1$.

All quantities below depend on parameters x and t. We assume that for the rest of the paper x and t are fixed.

Theorem 3.6. *(μ-perturbation in genus N)*
Consider a simple contour $\gamma_0 = \gamma(\mu_0)$ consisting of a finite union of oriented simple arcs $\gamma_0 = (\bigcup \gamma_{m,j}) \cup (\bigcup \gamma_{c,j})$ with the distinct arcs end points $\vec{\alpha}_0$ and depending on parameter μ (see Figure 1). Assume $\vec{\alpha}_0$ satisfies a system of equations

$$\vec{K}\,(\vec{\alpha}_0, \mu_0) = \vec{0},$$

and f is given by (15). Let $\gamma = \gamma(\vec{\alpha}, \mu)$ be the contour of a RH problem which seeks a function $h(z)$ which satisfies the following conditions

$$\begin{cases} h_+(z) + h_-(z) = 2W_j, \ \ on\ \gamma_{m,j}, \quad j = 0, 1, ..., N, \\ h_+(z) - h_-(z) = 2\Omega_j, \ \ on\ \gamma_{c,j}, \quad j = 1, ..., N, \\ h(z) + f(z) \ \ is\ analytic\ in\ \overline{\mathbb{C}} \backslash \gamma, \end{cases} \tag{43}$$

where $\Omega_j = \Omega_j(\vec{\alpha}, \mu)$ *and* $W_j = W_j(\vec{\alpha}, \mu)$ *are real constants whose numerical values will be determined from the RH conditions. Assume that there is a function* $h(z, \vec{\alpha}_0, \mu_0)$ *which satisfies (43) and suppose* $\frac{h'(z,\vec{\alpha}_0,\mu_0)}{R(z,\vec{\alpha}_0)} \neq 0$ *for all* z *on* γ*.*

Then the solution $\vec{\alpha} = \vec{\alpha}(\mu)$ *of the system*

$$\vec{K}\,(\vec{\alpha}, \mu) = \vec{0} \tag{44}$$

and $h(z, \vec{\alpha}(\mu), \mu)$ *which solves (43) are uniquely defined and smooth in* μ *in some neighborhood of* μ_0*.*

Moreover, $\Omega_j(\mu) = \Omega_j(\vec{\alpha}(\mu), \mu)$*, and* $W_j(\mu) = W_j(\vec{\alpha}(\mu), \mu)$ *are defined and smooth in* μ *in some neighborhood of* μ_0*.*

Furthermore

$$\frac{\partial \alpha_j}{\partial \mu}(\mu) = -\frac{2\pi i \frac{\partial K(\alpha_j, \vec{\alpha}, (\mu))}{\partial \mu}}{D(\vec{\alpha}(\mu), \mu) \oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta - \alpha_j(\mu)) R(\zeta, \vec{\alpha}(\mu))} d\zeta}, \tag{45}$$

$$\frac{\partial h}{\partial \mu}(z, \vec{\alpha}, \mu) = \frac{R(z, \vec{\alpha})}{2\pi i} \int_{\hat{\gamma}(\mu)} \frac{\frac{\partial f}{\partial \mu}(\zeta, \mu)}{(\zeta - z) R(\zeta, \vec{\alpha})} d\zeta, \tag{46}$$

where z *is inside of* $\hat{\gamma}$*,*

$$\frac{\partial \Omega_j}{\partial \mu}(\mu) = \frac{-1}{D} \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{m,N}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{c,j-1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,j-1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta & \dots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1} f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta \\ \oint_{\hat{\gamma}_{c,j+1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,j+1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{c,N}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \end{vmatrix}, \tag{47}$$

$$\frac{\partial W_j}{\partial \mu}(\mu) = \frac{-1}{D} \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{m,j-1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,j-1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta & \dots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1} f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta \\ \oint_{\hat{\gamma}_{m,j+1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,j+1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{m,N}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{m,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \dots & \dots & \dots \\ \oint_{\hat{\gamma}_{c,N}} \frac{d\zeta}{R(\zeta)} & \dots & \oint_{\hat{\gamma}_{c,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \end{vmatrix}, \tag{48}$$

where $R(\zeta) = R(\zeta, \vec{\alpha})$*.*

Proof. By Lemma 3.4 $\alpha_j(\mu)$ are continuously differentiable in μ. Formula for $\frac{\partial \alpha_j}{\partial \mu}$ is derived similarly as in [21]. We differentiate the modulation equations $K(\alpha_j) = K(\alpha_j, \vec{\alpha}, \mu) = 0$ which define $\vec{\alpha} = \vec{\alpha}(\mu)$ with respect to μ

$$\sum_{l=1}^{4N+1} \frac{\partial K(\alpha_j)}{\partial \alpha_l} \frac{\partial \alpha_l}{\partial \mu} + \frac{\partial K}{\partial \mu}(\alpha_j) = 0, \tag{49}$$

where the matrix $\left\{ \frac{\partial K(\alpha_j)}{\partial \alpha_l} \right\}_{j,l}$ is diagonal [21] so

$$\frac{\partial K(\alpha_j)}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \mu} = -\frac{\partial K(\alpha_j)}{\partial \mu}. \tag{50}$$

Since

$$\frac{\partial K(\alpha_j)}{\partial \alpha_j} = \frac{D(\vec{\alpha}, \mu)}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta, \mu)}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})} d\zeta \tag{51}$$

we arrive to the evolution equations for α_j:

$$\frac{\partial \alpha_j}{\partial \mu} = -\frac{2\pi i \frac{\partial K}{\partial \mu}(\alpha_j)}{D(\vec{\alpha}, \mu) \oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta,\mu)}{(\zeta - \alpha_j) R(\zeta, \vec{\alpha})} d\zeta}, \quad j = 0, ..., 4N+1. \tag{52}$$

Next we compute $\frac{\partial g}{\partial \mu}(z, \vec{\alpha}, \mu)$ which satisfies the scalar RHP

$$g_{\mu,+}(z) - g_{\mu,-}(z) = f_\mu(z), \quad z \in \gamma_{m,j}, \ j = 0, 1, ..., N. \tag{53}$$

Then

$$\frac{\partial g}{\partial \mu}(z, \vec{\alpha}, \mu) = \frac{R(z, \vec{\alpha})}{2\pi i} \int_{\hat{\gamma}} \frac{\frac{\partial f}{\partial \mu}(\zeta, \mu)}{(\zeta - z) R(\zeta, \vec{\alpha})} d\zeta \tag{54}$$

where z is outside of $\hat{\gamma}$, $\frac{\partial f}{\partial \mu}(z)$ behaves like $\log(z - z_0)$ near $z = z_0 = \frac{\mu}{2}$. So $\frac{\partial h}{\partial \mu}(z, \vec{\alpha}, \mu)$ satisfies (46) where z is inside of $\hat{\gamma}$.

Constants W_j and Ω_j are found from the linear system [21]

$$\oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta, \mu)}{R(\zeta, \vec{\alpha})} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{c,j}} \frac{\zeta^n \Omega_j}{R(\zeta, \vec{\alpha})} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{m,j}(\mu)} \frac{\zeta^n W_j}{R(\zeta, \vec{\alpha})} d\zeta = 0, n = 0, ...N-1. \tag{55}$$

Differentiating in μ and using Lemma 3.2 leads to

$$\oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f_\mu(\zeta, \mu)}{R(\zeta, \vec{\alpha})} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{c,j}} \frac{\zeta^n (\Omega_j)_\mu}{R(\zeta, \vec{\alpha})} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{m,j}(\mu)} \frac{\zeta^n (W_j)_\mu}{R(\zeta, \vec{\alpha})} d\zeta = 0, n = 0, ..., N-1 \tag{56}$$

or in matrix form

$$\begin{pmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,N}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \end{pmatrix}^T \begin{pmatrix} \frac{\partial \vec{W}}{\partial \beta_k} \\ \frac{\partial \vec{\Omega}}{\partial \beta_k} \end{pmatrix} = - \begin{pmatrix} \oint_{\hat{\gamma}(\mu)} \frac{f_\mu(\zeta,\mu)}{R(\zeta,\vec{\alpha})} d\zeta \\ \cdots \\ \oint_{\hat{\gamma}(\mu)} \frac{\zeta^{N-1} f_\mu(\zeta,\mu)}{R(\zeta,\vec{\alpha})} d\zeta \end{pmatrix}. \tag{57}$$

So $\frac{\partial \Omega_j}{\partial \beta_k}$ and $\frac{\partial W_j}{\partial \mu}$ satisfy (47) and (48). Note that $D \neq 0$ for distinct α_j's [21].

□

Remark 3.7. In [21] was considered the case when the contour γ was independent of external parameters x and t. We consider a case of the dependence on μ when the jump contour explicitly passes through $z = \frac{\mu}{2}$ a point of singularity of f.

Remark 3.8. The perturbation theorem 3.6 guarantees that the solution of the RHP (43) is uniquely continued with respect to external parameters. Additional sign conditions on $\Im h$ need to be satisfied, for h to correspond to an asymptotic solution of NLS as in [23]. The sign conditions have to be satisfied near γ and additionally on a semiinfinite complementary arcs connecting the arcs end points of γ to ∞.

3.3 Sign conditions and preservation of genus

The error estimates in the RH approach to semiclassical NLS require certain sign conditions to be satisfies. In particular $\Im h(z) = 0$ on $\gamma_{m,j}$ and $\Im h(z) \geq 0$ on $\gamma_{c,j}$.

On the real axis (in the limit from the upper half plane),

$$\Im f(z + i0) = \lim_{\delta \to 0^+} f(z + i\delta), \qquad z \in \mathbb{R}.$$

is a piecewise linear function [23]

$$\Im f(z + i0) = \begin{cases} \frac{\pi}{2}\left(\frac{\mu}{2} - |z|\right), & z < \frac{\mu}{2} \\ \frac{\pi}{2}\left(z - \frac{\mu}{2}\right), & z \geq \frac{\mu}{2} \end{cases}$$

and since $g(z)$ is real on the real axis, $\Im h(z + i0) = -\Im f(z + i0)$. It is important for us that the growth of $\Im h(z)$ can be estimated and be bounded from below away from zero as $z \to \infty$.

Similarly,

$$\Im f'(z + i0) = \begin{cases} \frac{\pi}{2} & z \leq 0, \\ -\frac{\pi}{2} & 0 < z \leq \frac{\mu}{2}, \\ \frac{\pi}{2} & z > \frac{\mu}{2}, \end{cases}$$

and since $g'(z)$ is real on the real axis, $\Im h'(z + i0) = -\Im f'(z + i0)$.

Definition 3.9. *Define $\gamma^\infty = \gamma^\infty(\vec{\alpha}, \mu)$ as an extension of a contour $\gamma(\vec{\alpha}, \mu) \in \Gamma(\vec{\alpha}, \mu)$ as $\gamma^\infty(\vec{\alpha}, \mu) = (\infty, \alpha_{4N}] \bigcup \gamma(\vec{\alpha}, \mu) \bigcup [\alpha_{4N+1}, \infty)$. Both additional arcs are considered as complementary arcs $\gamma_{c,0} = (\infty, \alpha_{4N}] \subset \mathbb{C}_-$, $\gamma_{c,N+1} = (\infty, \alpha_{4N}] \bigcup [\alpha_{4N+1}, \infty)$ and assume $\gamma_{c,N+1} = \overline{\gamma_{c,N+1}}$, so $\gamma^\infty = \overline{\gamma^\infty}$.*

Lemma 3.10. *If the conditions of Theorem 3.6 holds on $\gamma^\infty(\vec{\alpha}_0, \mu_0)$ for $\vec{K}(\vec{\alpha}_0, \mu_0) = \vec{0}$ then the statement of the theorem holds on $\gamma^\infty(\vec{\alpha}, \mu)$, where $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$.*

Proof.
The proof is unchanged since f is analytic near the additional semi-infinite arcs $\gamma_{c,0}$ and $\gamma_{c,N+1}$ and the jump condition for on the additional arcs $\gamma_{c,0}$ and $\gamma_{c,N+1}$ is taken to be zero ($\Omega_0 = 0$, $\Omega_{N+1} = 0$) [23]. □

Note that the conditions in Theorem 3.10 are more restrictive since $\gamma \subset \gamma^\infty$.

Definition 3.11. *A function h satisfies sign conditions on γ^∞ if $\Im h(z) = 0$ if $z \in \gamma_{m,j}$ and $\Im h(z) \geq 0$ if $z \in \gamma_{c,j}$ for all $j = 0, ..., N+1$. Denote $h \in SC(\gamma^\infty)$.*

Note that the sign conditions on $\gamma_{m,j}$ are satisfied automatically ($\Im h(z) = 0$) through the construction of $h(z)$ by (8) in the case of h solving a RHP (43). We only need to check the nonnegativity of $\Im h$ on the complementary arcs, especially on the semi-infinite arcs $(\infty, \alpha_{4N}]$ and $[\alpha_{4N+1}, \infty)$.

Theorem 3.12. *Let f be defined by (15). Let $\vec{K}(\vec{\alpha}_0, \mu_0) = \vec{0}$, $\gamma_0^\infty \in \Gamma(\vec{\alpha}_0, \mu_0)$ and assume h solves $RHP(\gamma_0^\infty, \vec{\alpha}_0, \mu_0, f)$ with $\frac{h'(z,\mu_0)}{R(z,\mu_0)} \neq 0$ for all $z \in \gamma_0^\infty$, and $h \in SC(\gamma_0^\infty)$.*

Then there is an open neighborhood of μ_0 where for all μ, there is an h which solves $RHP(\gamma^\infty, \vec{\alpha}, \mu, f)$ with $\gamma^\infty = \gamma^\infty(\vec{\alpha}, \mu)$, $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$, $\frac{h'(z,\mu)}{R(z,\mu)} \neq 0$ for all $z \in \gamma^\infty$, and $h \in SC(\gamma^\infty)$.

Proof.

Take any μ in a small enough open neighborhood of μ_0. There are two things we need to prove in addition to lemma 3.10: the sign conditions on γ^∞ and that $\frac{h'(z,\mu)}{R(z)} \neq 0$ on γ^∞.

Assume $h \in SC(\gamma_0^\infty)$. Then $h \in SC(\gamma(\mu))$ by continuity of h in z and μ and compactness of γ. The semi-infinite arcs $(\infty, \alpha_{4N}]$ and $[\alpha_{4N+1}, \infty)$ we can pushed to the real axis as $\big(-\infty + i0, -\frac{\mu}{2} + i0\big) \bigcup \big[-\frac{\mu}{2} + i0, \alpha_{4N}\big]$ and $\big(-\infty - i0, -\frac{\mu}{2} - i0\big) \bigcup \big[-\frac{\mu}{2} - i0, \alpha_{4N+1}\big]$ respectively. On $\big(-\infty - i0, -\frac{\mu}{2} - i0\big)$ and $\big(-\infty + i0, -\frac{\mu}{2} + i0\big)$, $\Im h(z)$ is positive and $\big[-\frac{\mu}{2}, \alpha_{4N}\big]$ and $\big[-\frac{\mu}{2}, \alpha_{4N+1}\big]$ are compact. So $\Im h \geq 0$ on $\gamma^\infty(\mu)$, that is $h \in SC(\gamma^\infty(\mu))$.

Let $\frac{h'(z,\mu_0)}{R(z,\vec{\alpha}(\mu_0))} \neq 0$ on γ_0^∞. Then there is a constant $C > 0$ such that $\left|\frac{h'(z,\mu_0)}{R(z,\vec{\alpha}(\mu_0))}\right| > C$ for all $z \in \gamma_0^\infty$. Consider the solution $h(z, \mu)$ of $RHP(\gamma^\infty, \vec{\alpha}, \mu)$, where $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$. By theorem 3.6 and lemma 3.10 such function exists and continuously differentiable in μ. Moreover, $h'(z, \mu)$ is continuous in μ. Since γ is a compact set in $\mathbb{C}$ and $\frac{h'(z,\mu)}{R(z,\vec{\alpha}(\mu))}$ is continuous in z and μ, we have $\frac{h'(z,\mu)}{R(z,\vec{\alpha}(\mu))} \neq 0$ for all $z \in \gamma$.

Now as before, the two semi-infinite pieces $(-\infty + i0, \alpha_{4N}]$ and $[\alpha_{4N+1}, -\infty - i0)$ are pushed to the real axis: $\big(-\infty + i0, -\frac{\mu}{2} + i0\big) \bigcup \big[-\frac{\mu}{2} + i0, \alpha_{4N}\big]$ and $\big(-\infty - i0, -\frac{\mu}{2} - i0\big) \bigcup \big[-\frac{\mu}{2} - i0, \alpha_{4N+1}\big]$ respectively. On $\big[-\frac{\mu}{2}, \alpha_{4N}\big]$ and $\big[-\frac{\mu}{2} - i0, \alpha_{4N+1}\big]$, $\frac{h'(z,\mu)}{R(z,\vec{\alpha}(\mu))} \neq 0$ by continuity on a compact set. Finally, for all $z \in \big(-\infty + i0, -\frac{\mu}{2} + i0\big)$, $\Im h'(z, \mu) = -\frac{\pi}{2}$ and $R(z, \vec{\alpha}) \in \mathbb{R}$. So $\frac{h'(z,\mu)}{R(z,\vec{\alpha}(\mu))} \neq 0$ for all $z \in \big(-\infty + i0, -\frac{\mu}{2} + i0\big)$. The interval $\big(-\infty - i0, -\frac{\mu}{2} - i0\big)$ is done similarly. So $\frac{h'(z,\mu)}{R(z,\vec{\alpha}(\mu))} \neq 0$ for all $z \in \gamma^\infty$, for any μ in a small neighborhood of μ_0.

□

Definition 3.13.
We define the (finite) genus $G = G(\mu)$ of the asymptotic solution of the semiclassical one dimensional focusing NLS with the initial condition defined through $f(z, \mu)$, as (finite) $N \in \mathbb{N}$ if there exists an asymptotic solution of NLS through the solution $h(z, \mu)$ of $RHP(\gamma^\infty, \vec{\alpha}, \mu, f)$ with $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{4N+1})$, such that $\frac{h'(z,\mu)}{R(z)} \neq 0$ for all $z \in \gamma^\infty$, and the signs conditions of h on γ^∞ are satisfied: $h \in SC(\gamma^\infty)$.

Remark 3.14. The definition of the genus of the asymptotic solution can alternatively be seen as the genus of the model (limiting) Riemann surface.

Theorem 3.15. *(Preservation of genus)*
Suppose for μ_0, the genus of the asymptotic solution of NLS with initial condition defined through $f(z,\mu_0)$ in (15), is $G(\mu_0)$.

Then there is an open neighborhood of μ_0 such that, for all μ in the neighborhood of μ_0, the genus of the asymptotic solution of NLS with initial condition defined through $f(z,\mu)$, is preserved $G(\mu) = G(\mu_0)$.

Proof. Follows from Theorem 3.12 and Definition 3.13. □

Corollary 3.16. *Fix x and $t > t_0$, where $t_0(x)$ is the time of the first break in the asymptotic solution. Then in some open neighborhood of $\mu = 2$ the genus of the solution is 2.*

Proof. For $\mu = 2$ and $t > t_0(x)$ the genus is 2 for all x [23]. By the Preservation of genus theorem 3.15, the genus is preserved in some open neighborhood of $\mu = 2$, including some open interval for $\mu < 2$. □

3.4 Computations

Figure 3 demonstrates comparison solutions of (70) and (63). The solutions are practically indistinguishable on the figure with the absolute difference less than 10^{-3} for $\mu \in [1, 4]$, which includes a critical value $\mu = 2$, the transition between (solitonless) pure radiation case ($\mu \geq 2$) and the region with solitons ($0 < \mu < 2$).

4 Appendix

4.1 Genus 0 region

It was shown in [23] that for all $\mu > 0$ and for all x, there is a breaking curve $t = t_0(x)$ in the (x,t) plane. The region $0 \leq t < t_0(x)$ has genus 0 in the sense of genus of the underlying Riemann surface for the square root

$$R(z) = \sqrt{(z-\alpha_0)(z-\alpha_1)},$$

where the branchcut is chosen along the main arc connecting α_0 and $\alpha_1 = \overline{\alpha}_0$, and the branch is fixed by $R(z) \to -z$ as $z \to +\infty$. The asymptotic solution of NLS is expressed in terms of $\alpha_0 = \alpha_0(x,t)$.

All expressions in the genus 0 region have simpler form. In particular:

$$h(z,\alpha_0,\mu) = \frac{R(z,\alpha_0)}{2\pi i} \oint_{\hat{\gamma}} \frac{f(\zeta,\mu)d\zeta}{(\zeta - z)R(\zeta,\alpha_0)}, \tag{58}$$

$$K(z,\alpha_0,\mu) = \frac{1}{2\pi i} \oint_{\hat{\gamma}} \frac{f(\zeta,\mu)d\zeta}{(\zeta - z)R(\zeta,\alpha_0)}, \tag{59}$$

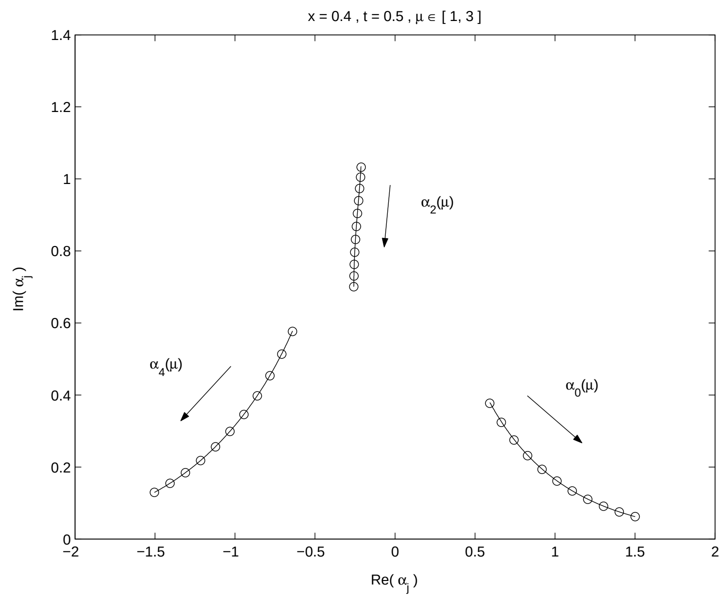


Figure 3: Comparison of μ evolution of $\vec{\alpha} = (\alpha_0, \alpha_2, \alpha_4)$ using (70) and (63).

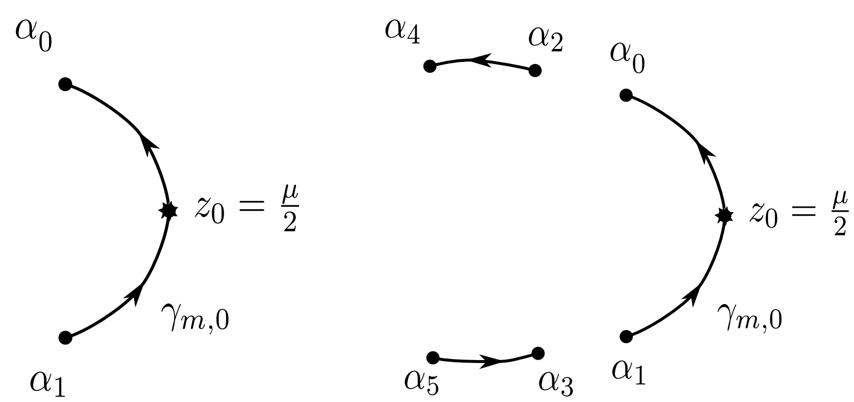


Figure 4: The jump contour in the case of genus 0 and genus 2 with complex conjugate symmetry in the notation of [23].

and with a slight abuse of notation

$$K(\alpha_0,\mu) := K(\alpha_0,\alpha_0,\mu) = \frac{1}{2\pi i}\oint_{\hat{\gamma}} \frac{f(\zeta,\mu)d\zeta}{(\zeta-\alpha_0)R(\zeta,\alpha_0)}, \tag{60}$$

and

$$\frac{\partial K}{\partial \mu}(\alpha_0,\mu) := \frac{\partial K}{\partial \mu}(\alpha_0,\alpha_0,\mu) = \frac{1}{2\pi i}\oint_{\hat{\gamma}} \frac{f_\mu(\zeta,\mu)d\zeta}{(\zeta-\alpha_0)R(\zeta,\alpha_0)} \tag{61}$$

Theorem 4.1. *(μ-perturbation in genus 0)*
Consider a simple contour $\gamma_0 = \gamma(\mu_0) = [\alpha_0(\mu_0), \overline{\alpha}_0(\mu_0)]$ an oriented simple arc with the distinct end points ($\alpha_0 \neq \overline{\alpha}_0$) and depending on parameter μ (see Figure 4). Assume $\alpha_0(\mu_0)$ satisfies the equation

$$K(\alpha_0,\mu_0) = 0,$$

and f is given by (15). Let $\gamma = \gamma(\vec{\alpha},\mu)$ be the contour of a RH problem which seeks a function $h(z)$ which satisfies the following conditions

$$\begin{cases} h_+(z) + h_-(z) = 0, \ \text{on } \gamma, \\ h(z) + f(z) \text{ is analytic in } \overline{\mathbb{C}}\backslash\gamma, \end{cases} \tag{62}$$

Assume that there is a function $h(z,\vec{\alpha}_0,\mu_0)$ which satisfies (62) and suppose $\frac{h'(z,\vec{\alpha}_0,\mu_0)}{R(z,\vec{\alpha}_0)} \neq 0$ for all z on γ.

Then the solution $\alpha_0(\mu)$ of the equation

$$K(\alpha_0,\mu) = 0 \tag{63}$$

and $h(z,\vec{\alpha}(\mu),\mu)$ which solves (62) are uniquely defined and smooth in μ in some neighborhood of μ_0.

Furthermore

$$\frac{\partial \alpha_0}{\partial \mu}(\mu) = -\frac{2\pi i \frac{\partial K}{\partial \mu}(\alpha_0,\mu)}{\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta-\alpha_0(\mu))R(\zeta,\alpha_0(\mu))} d\zeta}, \tag{64}$$

$$\frac{\partial h}{\partial \mu}(z,\alpha_0(\mu),\mu)=\frac{R(z,\alpha_0)}{2\pi i}\int_{\hat\gamma(\mu)}\frac{\frac{\partial f}{\partial \mu}(\zeta,\mu)}{(\zeta-z)R(\zeta,\alpha_0(\mu))}d\zeta, \tag{65}$$

where z is inside of $\hat\gamma$.

4.2 Genus 2 region

In the genus 2 region ($N=1$), with underlying Riemann surface for the square root

$$R(z)=\sqrt{(z-\alpha_0)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5)},$$

where the branchcut is chosen along the main arcs connecting α_0 and α_1, α_2 and α_4, α_5 and α_3; and the branch is fixed by $R(z)\to -z^3$ as $z\to+\infty$.

Taking into account the complex conjugate symmetry

$$\alpha_1=\overline{\alpha}_0,\;\; \alpha_3=\overline{\alpha}_2,\;\; \alpha_5=\overline{\alpha}_4. \tag{66}$$

$$h(z)=\frac{R(z)}{2\pi i}\left[\oint_{\hat\gamma}\frac{f(\zeta)}{(\zeta-z)R(\zeta)}d\zeta+\oint_{\hat\gamma_m}\frac{W}{(\zeta-z)R(\zeta)}d\zeta+\oint_{\hat\gamma_c}\frac{\Omega}{(\zeta-z)R(\zeta)}d\zeta\right], \tag{67}$$

where $\hat\gamma_m$ is a loop around the main arc $[\alpha_2,\alpha_4]\bigcup[\alpha_5,\alpha_3]$, and $\hat\gamma_c$ is a loop around the complementary arc $[\alpha_0,\alpha_2]\bigcup[\alpha_3,\alpha_1]$ (see Fig. 4).

Theorem 4.2. *(μ-perturbation in genus 2)*
Consider a simple contour $\gamma_0=\gamma(\mu_0)$ consisting of a finite union of oriented simple arcs $\gamma_0=(\bigcup\gamma_{m,j})\cup(\bigcup\gamma_{c,j})$ with the distinct arcs end points $\vec\alpha_0=(\alpha_0,\alpha_2,\alpha_4)$ and depending on parameter μ (see Figure 4). Assume $\vec\alpha_0$ satisfies a system of equations

$$\begin{cases} K(\alpha_0,\vec\alpha_0,\mu_0)=0,\\ K(\alpha_2,\vec\alpha_0,\mu_0)=0,\\ K(\alpha_4,\vec\alpha_0,\mu_0)=0,\end{cases}$$

and f is given by (15). Let $\gamma=\gamma(\vec\alpha,\mu)$ be the contour of a RH problem which seeks a function $h(z)$ which satisfies the following conditions

$$\begin{cases} h_+(z)+h_-(z)=0, \;\; on\ \gamma_{m,0},\\ h_+(z)+h_-(z)=2W, \;\; on\ \gamma_{m,1},\\ h_+(z)-h_-(z)=2\Omega, \;\; on\ \gamma_{c,1},\\ h(z)+f(z)\ is\ analytic\ in\ \overline{\mathbb{C}}\backslash\gamma,\end{cases} \tag{68}$$

where $\Omega=\Omega(\vec\alpha,\mu)$ and $W=W(\vec\alpha,\mu)$ are real constants whose numerical values will be determined from the RH conditions. Assume that there is a function $h(z,\vec\alpha_0,\mu_0)$ which satisfies (68) and suppose $\frac{h'(z,\vec\alpha_0,\mu_0)}{R(z,\vec\alpha_0)}\neq 0$ for all z on γ.

Then the solution $\vec\alpha=\vec\alpha(\mu)$ of the system

$$\begin{cases} K(\alpha_0,\vec\alpha_0,\mu)=0,\\ K(\alpha_2,\vec\alpha_0,\mu)=0,\\ K(\alpha_4,\vec\alpha_0,\mu)=0,\end{cases} \tag{69}$$

and $h(z, \vec{\alpha}(\mu), \mu)$ which solves (68) are uniquely defined and smooth in μ in some neighborhood of μ_0.

Moreover, $\Omega(\mu) = \Omega(\vec{\alpha}(\mu), \mu)$, and $W(\mu) = W(\vec{\alpha}(\mu), \mu)$ are defined and smooth in μ in some neighborhood of μ_0.

Furthermore

$$\frac{\partial \alpha_j}{\partial \mu}(x, t, \mu) = -\frac{2\pi i \frac{\partial K(\alpha_j, \vec{\alpha}, (\mu))}{\partial \mu}}{D(\vec{\alpha}(\mu), \mu) \oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta - \alpha_j(\mu)) R(\zeta, \vec{\alpha}(\mu))} d\zeta}, \tag{70}$$

$$\frac{\partial h}{\partial \mu}(z, \vec{\alpha}, x, t, \mu) = \frac{R(z, \vec{\alpha})}{2\pi i} \int_{\hat{\gamma}(\mu)} \frac{\frac{\partial f}{\partial \mu}(\zeta, \mu)}{(\zeta - z) R(\zeta, \vec{\alpha})} d\zeta, \tag{71}$$

where z is inside of $\hat{\gamma}$,

$$\frac{\partial \Omega}{\partial \mu}(x, t, \mu) = -\frac{1}{D} \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m} \frac{\zeta d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta & \oint_{\hat{\gamma}} \frac{\zeta f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta \end{vmatrix}, \tag{72}$$

$$\frac{\partial W}{\partial \mu}(x, t, \mu) = -\frac{1}{D} \begin{vmatrix} \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta & \oint_{\hat{\gamma}} \frac{\zeta f_\mu(\zeta)}{R(\zeta, \vec{\alpha})} d\zeta \\ \oint_{\hat{\gamma}_c} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c} \frac{\zeta d\zeta}{R(\zeta)} \end{vmatrix}, \tag{73}$$

where $\alpha_j = \alpha_j(x, t, \mu)$, $R(\zeta) = R(\zeta, \vec{\alpha}(x, t, \mu))$, $f(\zeta) = f(\zeta, x, t, \mu)$, $f_\mu(\zeta) = \frac{\partial f}{\partial \mu}(\zeta, x, t, \mu)$ and

$$D = D(x, t, \mu) = \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m} \frac{\zeta d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c} \frac{\zeta d\zeta}{R(\zeta)} \end{vmatrix}. \tag{74}$$

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